EDO STATE UNIVERSITY, IYAMHO

Department of Mathematics

MTH 121: Vectors, Geometry and Dynamics

Full Lecture Notes with Worked Examples, Diagrams, Assignments and Model Solutions

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These notes are prepared for undergraduate teaching and revision in vector algebra, coordinate geometry and elementary dynamics.

Contents

Course Outline and Learning Outcomes

Course Outline

1. Types of vectors: points, line and relative vectors; geometrical representation of vectors in one, two and three dimensions.
2. Addition of vectors and multiplication by a scalar; components of vectors in one, two and three dimensions; direction cosines.
3. Linear combinations, span, linear dependence and independence of vectors.
4. Scalar and vector products of two vectors and applications to angles, projections, areas and normals.
5. Two-dimensional coordinate geometry: straight lines, point of division of a line, distance between two points and angle between two lines.
6. Equation of circle, tangent and normal to a circle; elementary properties of parabola, ellipse and hyperbola.
7. Kinematics of a particle; force, momentum and laws of motion.
8. Motion under gravity, projectiles, angular momentum and conservation of angular momentum.
9. Work, energy and power; simple harmonic motion; rigid body ideas; triangle and parallelogram of forces; equilibrium of coplanar forces; couples and centre of gravity.

Learning Outcomes

At the end of this course, students should be able to define scalar and vector quantities, represent vectors geometrically and algebraically, compute vector products, use vector methods in coordinate geometry, analyze elementary dynamics problems and solve standard examination questions in vectors, geometry and dynamics.

Chapter 1

Scalars, Vectors and Vector Representation

1.1 Scalar and Vector Quantities

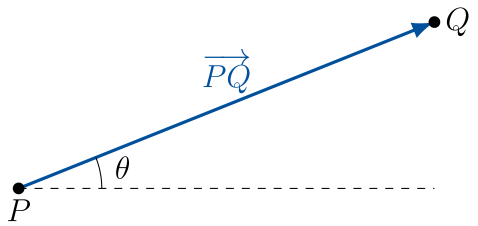
Definition 1.1. *A scalar quantity is a quantity completely described by magnitude only. Examples include mass, time, length, area, volume, temperature, distance, speed and energy.*

Definition 1.2. *A vector quantity is a quantity completely described by both magnitude and direction. Examples include displacement, velocity, acceleration, force, momentum and torque.*

For example, saying that a student walked 5 km only gives a distance. But saying the student walked 5 km due east gives a displacement, because both magnitude and direction are specified.

1.2 Geometrical Representation of a Vector

A vector is represented by a directed line segment. If the tail is at P and the head is at Q, the vector is written as $\overrightarrow{PQ}$. Its length gives magnitude while its arrow gives direction.



If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then

$$\overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

The magnitude is

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

1.3 Types of Vectors

1. **Unit vector:** a vector of magnitude 1. If $\mathbf{a} \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{a}$ is
$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$
2. **Zero vector:** a vector of magnitude zero, denoted by $\mathbf{0}$.
3. **Free vector:** a vector whose position is not fixed; it may be moved parallel to itself.
4. **Localized vector:** a vector whose point of application is fixed, such as a force acting at a definite point.
5. **Equal vectors:** vectors having the same magnitude and direction.
6. **Parallel vectors:** vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} = \lambda\mathbf{b}$ for some scalar λ.
7. **Like and unlike vectors:** like vectors have the same direction; unlike vectors have opposite directions.
8. **Coplanar vectors:** vectors lying in the same plane.
9. **Concurrent vectors:** vectors whose lines of action pass through one point.

Example 1.1. *Find the vector from $A(1, 2, -1)$ to $B(-3, 1, 2)$ and its magnitude.*

Solution.
$$\overrightarrow{AB} = (-3-1)\mathbf{i} + (1-2)\mathbf{j} + (2-(-1))\mathbf{k} = -4\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$
Hence
$$|\overrightarrow{AB}| = \sqrt{(-4)^2 + (-1)^2 + 3^2} = \sqrt{16+1+9} = \sqrt{26}.$$
□

Example 1.2. *Find a unit vector in the direction of $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.*

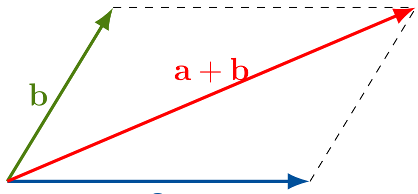
Solution.
$$|\mathbf{a}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}.$$
Therefore
$$\widehat{\mathbf{a}} = \frac{3}{\sqrt{14}}\mathbf{i} - \frac{1}{\sqrt{14}}\mathbf{j} + \frac{2}{\sqrt{14}}\mathbf{k}.$$
□

Chapter 2

Vector Addition, Subtraction and Scalar Multiplication

2.1 Addition of Vectors

Vector addition may be performed geometrically by the triangle law or parallelogram law.



Parallelogram law

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$$

2.2 Subtraction of Vectors

The difference $\mathbf{a} - \mathbf{b}$ is obtained by adding $\mathbf{a}$ to the negative of $\mathbf{b}$:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

2.3 Scalar Multiplication

If k is a scalar and $\mathbf{a}$ is a vector, then

$$k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j} + ka_3\mathbf{k}.$$

If $k > 0$, $k\mathbf{a}$ has the same direction as $\mathbf{a}$. If $k < 0$, $k\mathbf{a}$ has the opposite direction.

Theorem 2.1 (Algebra of Vector Addition)**.** *For vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ *and scalars* $m, n \in$

$\mathbb{R}$,

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a}, \\ (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \mathbf{a} + (\mathbf{b} + \mathbf{c}), \\ m(\mathbf{a} + \mathbf{b}) &= m\mathbf{a} + m\mathbf{b}, \\ (m + n)\mathbf{a} &= m\mathbf{a} + n\mathbf{a}. \end{aligned}$$

Proof. The proof follows by comparing the corresponding components of the vectors on the left and the right hand sides. For instance, if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (b_1 + a_1, b_2 + a_2, b_3 + a_3) = \mathbf{b} + \mathbf{a}.$$

The other identities are proved similarly. □

Example 2.1. *Let* $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ *and* $\mathbf{b} = -\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$. *Find* $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ *and* $3\mathbf{a} - 2\mathbf{b}$.

Solution.

$$\mathbf{a} + \mathbf{b} = (2 - 1)\mathbf{i} + (-3 + 5)\mathbf{j} + (4 - 2)\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

$$\mathbf{a} - \mathbf{b} = (2 + 1)\mathbf{i} + (-3 - 5)\mathbf{j} + (4 + 2)\mathbf{k} = 3\mathbf{i} - 8\mathbf{j} + 6\mathbf{k}.$$

Also,

$$3\mathbf{a} = 6\mathbf{i} - 9\mathbf{j} + 12\mathbf{k}, \quad 2\mathbf{b} = -2\mathbf{i} + 10\mathbf{j} - 4\mathbf{k}.$$

Therefore

$$3\mathbf{a} - 2\mathbf{b} = (6 + 2)\mathbf{i} + (-9 - 10)\mathbf{j} + (12 + 4)\mathbf{k} = 8\mathbf{i} - 19\mathbf{j} + 16\mathbf{k}.$$

□

Chapter 3

Linear Combinations, Span and Linear Independence

3.1 Linear Combination of Vectors

Definition 3.1. *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in* $\mathbb{R}^m$*. A vector* $\mathbf{w}$ *is called a linear combination of* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *if there exist scalars* $c_1, c_2, \ldots, c_n$ *such that*

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Linear combinations are fundamental because they describe how new vectors are formed from given vectors by scaling and adding.

Example 3.1. *Show that* $\mathbf{w} = (7, 1, 5)$ *is a linear combination of* $\mathbf{u} = (1, 1, 0)$*,* $\mathbf{v} = (2, -1, 1)$ *and* $\mathbf{r} = (0, 1, 2)$*.*

Solution. We seek scalars a, b, c such that

$$a(1, 1, 0) + b(2, -1, 1) + c(0, 1, 2) = (7, 1, 5).$$

Equating components gives

$$a + 2b = 7, \qquad a - b + c = 1, \qquad b + 2c = 5.$$

From $a = 7 - 2b$ and $c = (5 - b)/2$. Substitute in the second equation:

$$7 - 2b - b + \frac{5 - b}{2} = 1.$$

Multiplying by 2 gives

$$14 - 6b + 5 - b = 2 \Rightarrow 19 - 7b = 2 \Rightarrow b = \frac{17}{7}.$$

Then

$$a = 7 - \frac{34}{7} = \frac{15}{7}, \qquad c = \frac{5 - 17/7}{2} = \frac{9}{7}.$$

Hence

$$\mathbf{w} = \frac{15}{7}\mathbf{u} + \frac{17}{7}\mathbf{v} + \frac{9}{7}\mathbf{r}.$$

□

3.2 Span

Definition 3.2. *The span of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is the set of all their linear combinations:*

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n : c_i \in \mathbb{R}\}.$$

If two non-parallel vectors lie in a plane, their span is that plane. If three non-coplanar vectors are in space, their span is $\mathbb{R}^3$.

3.3 Linear Dependence and Independence

Definition 3.3. *Vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly dependent if there exist scalars* $c_1, \ldots, c_n$*, not all zero, such that*

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

They are linearly independent if the only solution is $c_1 = c_2 = \cdots = c_n = 0$.

Theorem 3.1. *Three vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in* $\mathbb{R}^3$ *are linearly independent if and only if the determinant*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

is non-zero.

Example 3.2. *Determine whether the vectors* $\mathbf{a} = (1, 2, 1)$*,* $\mathbf{b} = (2, 1, 0)$ *and* $\mathbf{c} = (3, 5, 2)$ *are linearly independent.*

Solution. Compute

$$\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 3 & 5 & 2 \end{pmatrix} = 1\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}.$$

Thus

$$= 1(2) - 2(4) + (10 - 3) = 2 - 8 + 7 = 1 \neq 0.$$

Therefore the vectors are linearly independent. □

Example 3.3. *Express* $\mathbf{w} = (5, 4)$ *as a linear combination of* $\mathbf{u} = (1, 2)$ *and* $\mathbf{v} = (3, -1)$.

Solution. Let $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. Then

$$a(1, 2) + b(3, -1) = (5, 4).$$

Equating components,

$$a + 3b = 5, \qquad 2a - b = 4.$$

From $a = 5 - 3b$. Substitute:

$$2(5 - 3b) - b = 4 \Rightarrow 10 - 7b = 4 \Rightarrow b = \frac{6}{7}.$$

Then

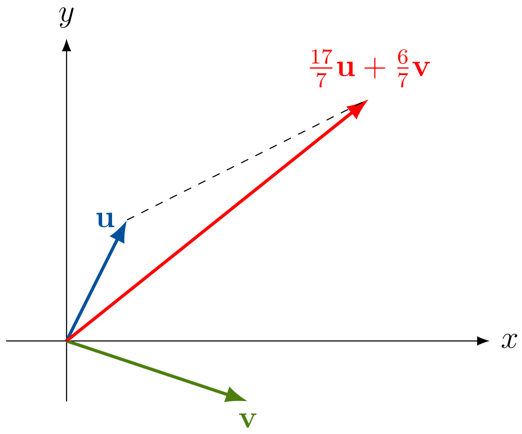
$$a = 5 - \frac{18}{7} = \frac{17}{7}.$$

So

$$(5, 4) = \frac{17}{7}(1, 2) + \frac{6}{7}(3, -1).$$

□

3.4 Geometrical Meaning of Linear Combination



Linear combinations provide algebraic descriptions of points, lines and planes.

Chapter 4

Components, Direction Ratios and Direction Cosines

4.1 Components of a Vector

In three dimensions,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The quantities x, y, z are the rectangular components of $\mathbf{r}$.

4.2 Direction Ratios and Direction Cosines

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then x, y, z are direction ratios. If $|\mathbf{r}| = r$, the direction cosines are

$$l = \frac{x}{r}, \qquad m = \frac{y}{r}, \qquad n = \frac{z}{r}.$$

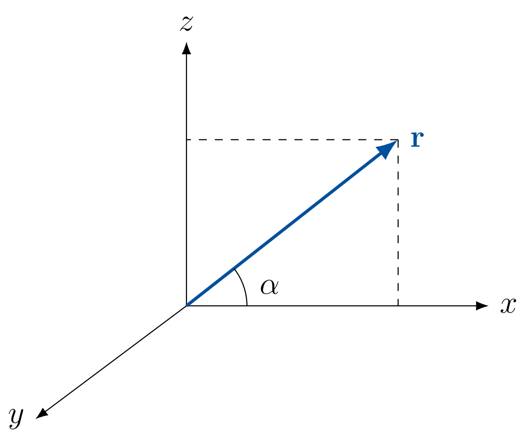
They satisfy

$$l^2 + m^2 + n^2 = 1.$$

Proof. Since $r = \sqrt{x^2 + y^2 + z^2}$,

$$l^2 + m^2 + n^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2 + y^2 + z^2}{r^2} = 1.$$

□



Example 4.1. *Find the magnitude and direction cosines of* $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$.

Solution.

$$|\mathbf{a}| = \sqrt{3^2 + 7^2 + (-4)^2} = \sqrt{9 + 49 + 16} = \sqrt{74}.$$

Therefore the direction cosines are

$$l = \frac{3}{\sqrt{74}}, \quad m = \frac{7}{\sqrt{74}}, \quad n = -\frac{4}{\sqrt{74}}.$$

□

Example 4.2. *Find the vector of magnitude* 5 *in the direction of* $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Solution. Let $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Then $|\mathbf{a}| = \sqrt{16 + 9 + 1} = \sqrt{26}$. The unit vector is

$$\widehat{\mathbf{a}} = \frac{4\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{26}}.$$

Hence the required vector is

$$5\widehat{\mathbf{a}} = \frac{20}{\sqrt{26}}\mathbf{i} - \frac{15}{\sqrt{26}}\mathbf{j} + \frac{5}{\sqrt{26}}\mathbf{k}.$$

□

Chapter 5

The Scalar or Dot Product

5.1 Definition and Basic Properties

Definition 5.1. *The scalar product of two non-zero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta,$$

where θ *is the angle between them.*

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Theorem 5.1. *For vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and scalar* m*:*

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
3. $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b})$.
4. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

5.2 Perpendicular Vectors

Two non-zero vectors are perpendicular if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

This follows because $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$, and $\cos \theta = 0$ when $\theta = 90°$.

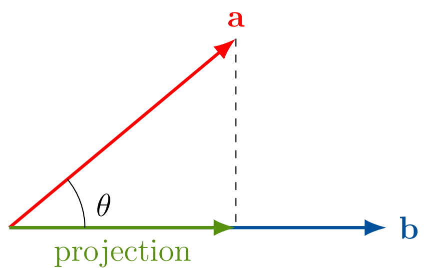
5.3 Projection of a Vector

The scalar projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$\operatorname{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}.$$

The vector projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}.$$



Example 5.1. *If* $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ *and* $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$*, find* $\mathbf{a} \cdot \mathbf{b}$*.*

Solution.

$$\mathbf{a} \cdot \mathbf{b} = (3)(-2) + (4)(3) + (-1)(1) = -6 + 12 - 1 = 5.$$

□

Example 5.2. *Find the angle between* $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ *and* $\mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$*.*

Solution.

$$\mathbf{a} \cdot \mathbf{b} = (3)(6) + (2)(-3) + (-1)(2) = 18 - 6 - 2 = 10.$$

$$|\mathbf{a}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}, \quad |\mathbf{b}| = \sqrt{6^2 + (-3)^2 + 2^2} = 7.$$

Thus

$$\cos\theta = \frac{10}{7\sqrt{14}}, \qquad \theta = \cos^{-1}\left(\frac{10}{7\sqrt{14}}\right).$$

□

Example 5.3. *Find* λ *such that* $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ *and* $3\mathbf{i} + 2\lambda\mathbf{j}$ *are perpendicular.*

Solution. Perpendicularity means dot product is zero:

$$(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (3\mathbf{i} + 2\lambda\mathbf{j} + 0\mathbf{k}) = 6 - 2\lambda = 0.$$

Hence $\lambda = 3$. □

Chapter 6

The Vector or Cross Product

6.1 Definition

Definition 6.1. *The vector product of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is a vector defined by*

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \mathbf{n},$$

where $\mathbf{n}$ *is a unit vector perpendicular to the plane of* $\mathbf{a}$ *and* $\mathbf{b}$*, chosen according to the right-hand rule.*

The cross product is not commutative:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}).$$

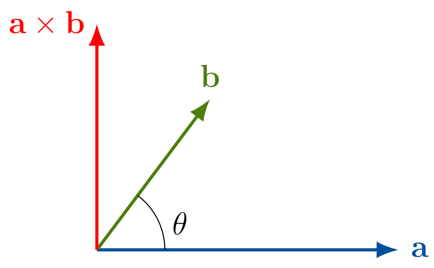
6.2 Determinant Form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Thus

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$



6.3 Area Applications

The area of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is

$$A = |\mathbf{a} \times \mathbf{b}|.$$

The area of the triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is

$$A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|.$$

Example 6.1. *If* $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ *and* $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$*, find* $\mathbf{a} \times \mathbf{b}$*.*

Solution.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}.$$

Expanding,

$$= \mathbf{i}[(-3)(-2) - (-1)(4)] - \mathbf{j}[(2)(-2) - (-1)(1)] + \mathbf{k}[(2)(4) - (-3)(1)].$$

Therefore

$$\mathbf{a} \times \mathbf{b} = 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}.$$

□

Example 6.2. *Find the area of the triangle with vertices* $P(1, 3, 2)$*,* $Q(2, -1, 1)$ *and* $R(-1, 2, 3)$*.*

Solution.

$$\overrightarrow{PQ} = (1, -4, -1), \qquad \overrightarrow{PR} = (-2, -1, 1).$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}.$$

$$= \mathbf{i}[(-4)(1) - (-1)(-1)] - \mathbf{j}[(1)(1) - (-1)(-2)] + \mathbf{k}[(1)(-1) - (-4)(-2)].$$

$$= -5\mathbf{i} + \mathbf{j} - 9\mathbf{k}.$$

Hence

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{25 + 1 + 81} = \sqrt{107}.$$

Area of triangle is

$$\frac{1}{2}\sqrt{107}.$$

□

Example 6.3. *Determine a unit vector perpendicular to the plane of* $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ *and* $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$*.*

Solution. A perpendicular vector is $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}.$$

Its magnitude is

$$\sqrt{15^2 + (-10)^2 + 30^2} = \sqrt{1225} = 35.$$

So a unit normal vector is

$$\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The opposite unit normal is also valid. □

Chapter 7

Scalar Triple Product and Vector Triple Product

7.1 Scalar Triple Product

Definition 7.1. *The scalar triple product of* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *is*

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

In component form,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Its absolute value gives the volume of the parallelepiped determined by the three vectors.

7.2 Coplanarity Test

Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar if and only if

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0.$$

Example 7.1. *Evaluate* $(2\mathbf{i} - 3\mathbf{j}) \cdot [(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - \mathbf{k})]$.

Solution. Using determinant form,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}.$$

Expanding along the first row,

$$= 2\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = 2(-1) + 3[(-1) - (-3)] = -2 + 6 = 4.$$

□

7.3 Vector Triple Product

The vector triple product is written as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

A useful identity is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Note that cross product is not associative in general:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

Example 7.2. *Let* $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ *and* $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. *Compare* $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ *and* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Solution. First,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}.$$

Then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (-\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 24\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}.$$

Next,

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = 0\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}.$$

Thus

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (-5\mathbf{j} - 5\mathbf{k}) = 15\mathbf{i} + 5\mathbf{j} - 15\mathbf{k}.$$

The two results are not equal. Hence the cross product is not associative. □

Chapter 8

Two-Dimensional Coordinate Geometry

8.1 Distance Between Two Points

For $A(x_1, y_1)$ and $B(x_2, y_2)$,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

8.2 Midpoint and Section Formula

The midpoint of $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

If P divides AB internally in the ratio $m : n$, then

$$P = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right).$$

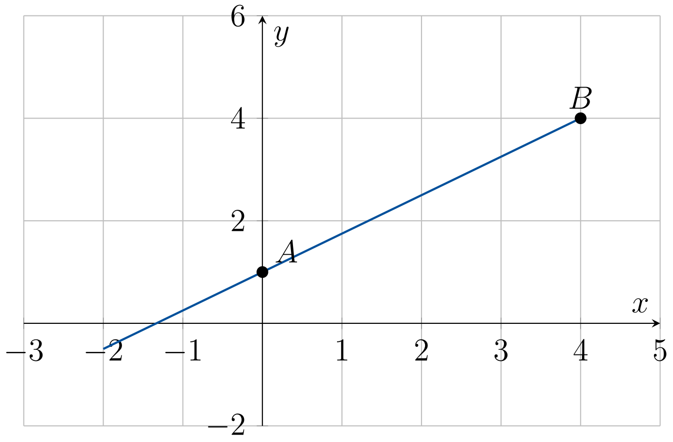
8.3 Straight Lines

The slope of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Forms of a straight line include

$$y - y_1 = m(x - x_1), \quad y = mx + c, \quad ax + by + c = 0.$$



8.4 Angle Between Two Lines

If two lines have slopes m_1 and m_2, the angle θ between them is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

If $m_1 m_2 = -1$, the lines are perpendicular.

Example 8.1. *Find the equation of the line through* $(2, -1)$ *and* $(6, 7)$*.*

Solution. The slope is

$$m = \frac{7-(-1)}{6-2} = \frac{8}{4} = 2.$$

Using point-slope form,

$$y + 1 = 2(x - 2),$$

so

$$y = 2x - 5.$$

□

Example 8.2. *Find the point that divides the line joining* $A(1, 2)$ *and* $B(7, 8)$ *internally in the ratio* $2 : 1$*.*

Solution.

$$P = \left(\frac{2(7)+1(1)}{2+1}, \frac{2(8)+1(2)}{2+1}\right) = \left(\frac{15}{3}, \frac{18}{3}\right) = (5, 6).$$

□

Chapter 9

Circle, Tangent and Normal

9.1 Equation of a Circle

A circle with centre (a, b) and radius r has equation

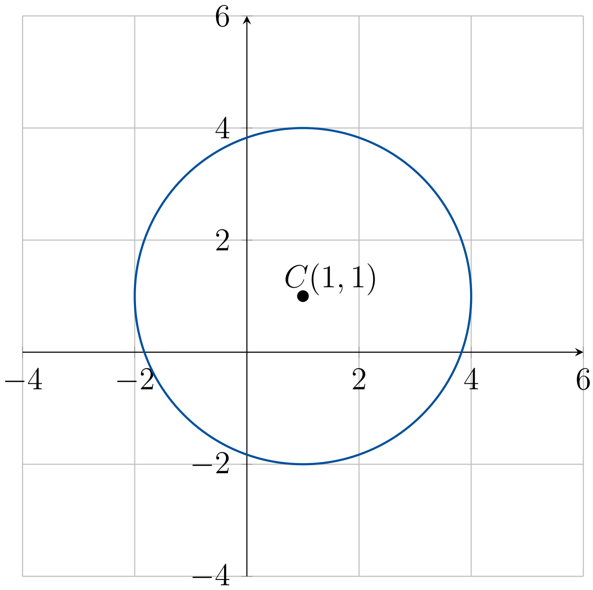
$$(x-a)^2 + (y-b)^2 = r^2.$$

The general form is

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

with centre $(-g, -f)$ and radius

$$r = \sqrt{g^2 + f^2 - c}.$$



9.2 Tangent and Normal to a Circle

For the circle $x^2 + y^2 = r^2$, the tangent at (x_1, y_1) is

$$xx_1 + yy_1 = r^2.$$

The normal is the line through $(0, 0)$ and (x_1, y_1).

Example 9.1. *Find the centre and radius of* $x^2 + y^2 - 4x + 6y - 12 = 0$.

Solution. Complete the square:

$$x^2 - 4x + y^2 + 6y = 12.$$

$$(x-2)^2 - 4 + (y+3)^2 - 9 = 12.$$

Hence

$$(x-2)^2 + (y+3)^2 = 25.$$

The centre is $(2, -3)$ and the radius is 5. $\square$

Chapter 10

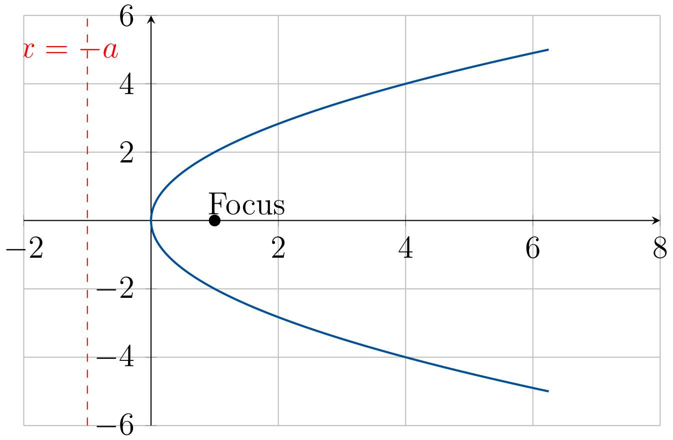
Conic Sections: Parabola, Ellipse and Hyperbola

10.1 Parabola

A parabola is the locus of a point that moves so that its distance from a fixed point called the focus equals its distance from a fixed line called the directrix. The standard equation is

$$y^2 = 4ax.$$

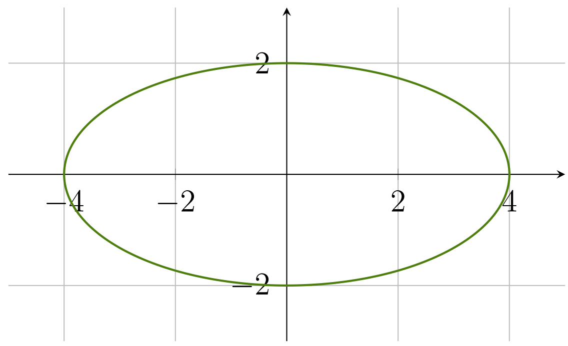
Its focus is $(a, 0)$ and directrix is $x = -a$.



10.2 Ellipse

The standard equation of an ellipse centered at the origin is

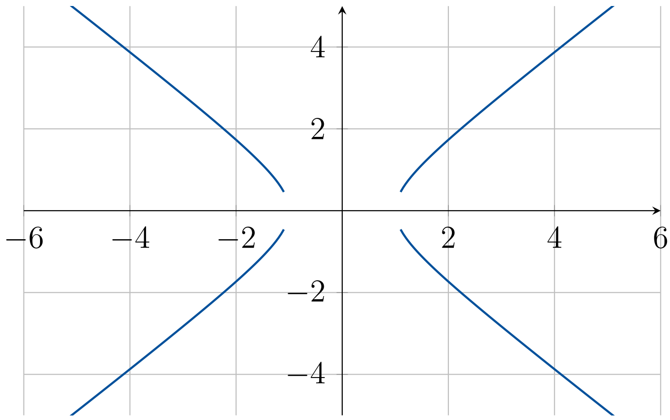
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



10.3 Hyperbola

The standard equation of a hyperbola centered at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



Example 10.1. *Find the focus and directrix of $y^2 = 12x$.*

Solution. Compare with $y^2 = 4ax$. Then $4a = 12$, so $a = 3$. The focus is $(3, 0)$ and the directrix is $x = -3$. $\square$

Chapter 11

Kinematics of a Particle

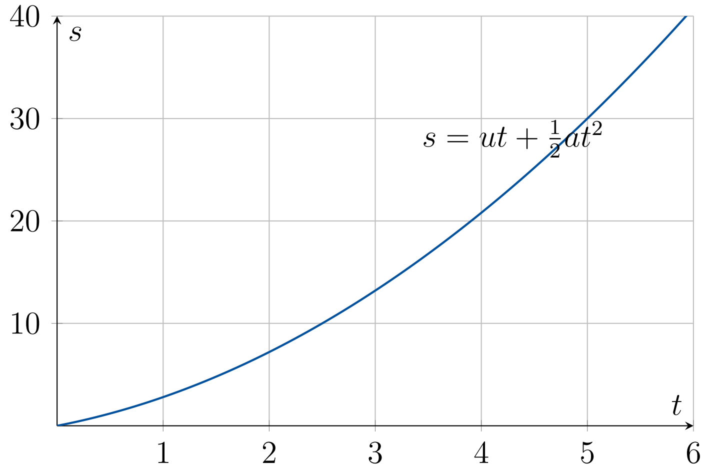
11.1 Basic Quantities

Kinematics studies motion without considering the causes of motion. The key quantities are displacement, velocity and acceleration. If $s(t)$ is displacement at time t, then

$$v(t) = \frac{ds}{dt}, \qquad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

For uniform acceleration,

$$v = u + at, \quad s = ut + \frac{1}{2}at^2, \quad v^2 = u^2 + 2as.$$



Example 11.1. *A particle starts from rest and moves with constant acceleration* $3\,m/s^2$*. Find its velocity and distance after 8 seconds.*

Solution. Here $u = 0$, $a = 3$ and $t = 8$. Thus

$$v = u + at = 0 + 3(8) = 24\,\text{m/s}.$$

Also,

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}(3)(8^2) = 96\,\text{m}.$$

□

Chapter 12

Motion Under Gravity and Projectiles

12.1 Vertical Motion Under Gravity

Near the surface of the earth, acceleration due to gravity is approximately $g = 9.8\,\mathrm{m/s}^2$ downward. Taking upward as positive,

$$v = u - gt, \quad s = ut - \frac{1}{2}gt^2, \quad v^2 = u^2 - 2gs.$$

12.2 Projectile Motion

If a particle is projected with speed u at angle θ to the horizontal, then

$$x = u\cos\theta\, t, \quad y = u\sin\theta\, t - \frac{1}{2}gt^2.$$

Eliminating t gives the trajectory

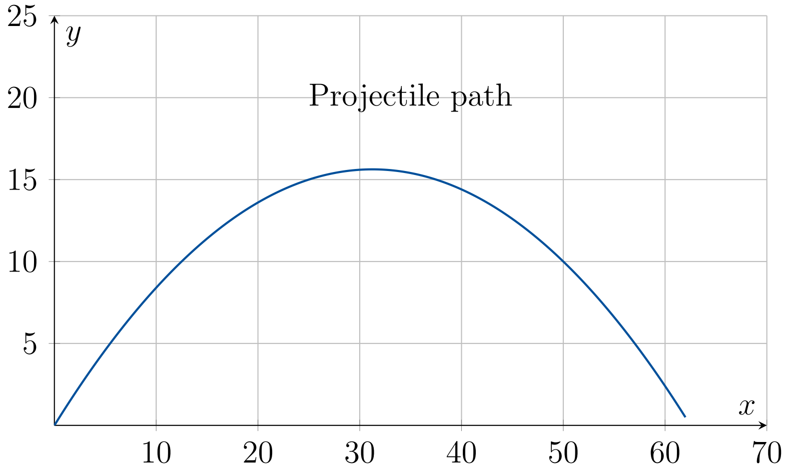
$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}.$$

The range is

$$R = \frac{u^2\sin 2\theta}{g},$$

and the maximum height is

$$H = \frac{u^2\sin^2\theta}{2g}.$$



Example 12.1. *A projectile is fired with speed* $20\,m/s$ *at* 30° *to the horizontal. Take* $g = 10\,m/s^2$. *Find the time of flight, maximum height and range.*

Solution. Time of flight:

$$T = \frac{2u\sin\theta}{g} = \frac{2(20)\sin 30^\circ}{10} = \frac{40(1/2)}{10} = 2\,\text{s}.$$

Maximum height:

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{400(1/4)}{20} = 5\,\text{m}.$$

Range:

$$R = \frac{u^2\sin 2\theta}{g} = \frac{400\sin 60^\circ}{10} = 40\cdot\frac{\sqrt{3}}{2} = 20\sqrt{3}\,\text{m}.$$

□

Chapter 13

Force, Momentum and Newton's Laws

13.1 Newton's Laws of Motion

Newton's second law states that the resultant force is proportional to the rate of change of momentum. For constant mass,

$$\mathbf{F} = m\mathbf{a}.$$

Momentum is

$$\mathbf{p} = m\mathbf{v}.$$

Impulse is the change in momentum:

$$\mathbf{J} = \Delta\mathbf{p}.$$

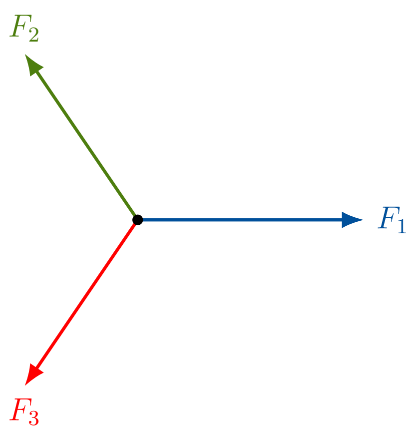
13.2 Equilibrium of Forces

For a particle in equilibrium,

$$\sum \mathbf{F} = \mathbf{0}.$$

In two dimensions,

$$\sum F_x = 0, \qquad \sum F_y = 0.$$



Example 13.1. *A body of mass 5 kg accelerates at* $4\,m/s^2$*. Find the resultant force.*

Solution.

$$F = ma = 5(4) = 20\,\mathrm{N}.$$

□

Chapter 14

Work, Energy and Power

14.1 Work Done

Work done by a constant force $\mathbf{F}$ in moving a body through displacement $\mathbf{s}$ is

$$W = \mathbf{F} \cdot \mathbf{s} = F s \cos\theta.$$

14.2 Energy

Kinetic energy is

$$K = \frac{1}{2}mv^2.$$

Potential energy near the earth's surface is

$$U = mgh.$$

14.3 Power

Power is the rate of doing work:

$$P = \frac{W}{t} = \mathbf{F} \cdot \mathbf{v}.$$

Example 14.1. *A force* $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ *moves a particle from* $A(1,2)$ *to* $B(5,5)$*. Find the work done.*

Solution. The displacement is

$$\mathbf{s} = (5-1)\mathbf{i} + (5-2)\mathbf{j} = 4\mathbf{i} + 3\mathbf{j}.$$

Thus

$$W = \mathbf{F} \cdot \mathbf{s} = (3)(4) + (4)(3) = 12 + 12 = 24\,\mathrm{J}.$$

□

Chapter 15

Simple Harmonic Motion

15.1 Definition

A particle moves with simple harmonic motion if its acceleration is directly proportional to its displacement from a fixed point and directed towards that point:

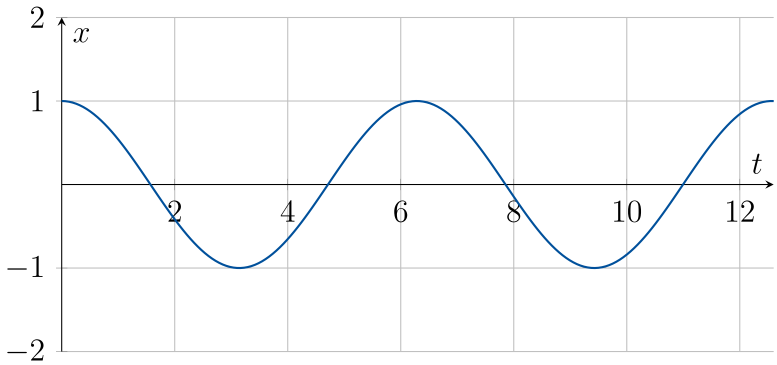
$$a = -\omega^2 x.$$

The general solution is

$$x = A\cos(\omega t) + B\sin(\omega t).$$

The period is

$$T = \frac{2\pi}{\omega}.$$



Example 15.1. *A particle executes SHM with $x = 5\cos(2t)$. Find its amplitude, angular frequency and period.*

Solution. The amplitude is 5. The angular frequency is $\omega = 2$. Hence

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$$

□

Chapter 16

Rigid Body, Couples and Centre of Gravity

16.1 Moment of a Force

The moment of a force about a point is

$$\text{Moment} = Fd,$$

where d is the perpendicular distance from the point to the line of action of the force. In vector form,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}.$$

16.2 Couple

A couple consists of two equal, opposite and parallel forces whose lines of action are different. The moment of a couple is independent of the point about which moments are taken.

16.3 Centre of Gravity

If particles of masses $m_1, \ldots, m_n$ are placed at positions $\mathbf{r}_1, \ldots, \mathbf{r}_n$, the centre of gravity is

$$\bar{\mathbf{r}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}.$$

Example 16.1. *Masses* $2, 3, 5$ *kg are placed at* $x = 1, 4, 7$ *respectively. Find the centre of gravity on the line.*

Solution.

$$\bar{x} = \frac{2(1) + 3(4) + 5(7)}{2 + 3 + 5} = \frac{2 + 12 + 35}{10} = \frac{49}{10} = 4.9.$$

□

Chapter 17

Past-Question Style Worked Problems

17.1 Vectors and Linear Combination

Example 17.1. *Given* $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ *and* $\mathbf{c} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, *compute* $(2\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$.

Solution.

$$2\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}.$$

$$2\mathbf{a} + \mathbf{b} = (6+2)\mathbf{i} + (-2+1)\mathbf{j} + (8-1)\mathbf{k} = 8\mathbf{i} - \mathbf{j} + 7\mathbf{k}.$$

Therefore

$$(2\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = (8)(1) + (-1)(3) + (7)(2) = 8 - 3 + 14 = 19.$$

□

Example 17.2. *Find* x *so that* $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ *and* $\mathbf{b} = 2\mathbf{i} + 6\mathbf{j} + x\mathbf{k}$ *are perpendicular.*

Solution.

$$\mathbf{a} \cdot \mathbf{b} = 2(2) + 4(6) + (-7)x = 4 + 24 - 7x = 28 - 7x.$$

For perpendicular vectors, $\mathbf{a} \cdot \mathbf{b} = 0$, so

$$28 - 7x = 0 \Rightarrow x = 4.$$

□

Example 17.3. *If* $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ *and* $\mathbf{b} = \mathbf{j} + 4\mathbf{k}$, *find the component of* $\mathbf{a}$ *along* $\mathbf{b}$.

Solution. The scalar component is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}.$$

Now

$$\mathbf{a} \cdot \mathbf{b} = (2)(0) + (-3)(1) + (4)(4) = 13, \quad |\mathbf{b}| = \sqrt{1^2 + 4^2} = \sqrt{17}.$$

Therefore the component of $\mathbf{a}$ along $\mathbf{b}$ is

$$\frac{13}{\sqrt{17}}.$$

□

17.2 Cross Product and Area

Example 17.4. *Find the area of the parallelogram whose adjacent sides are* $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ *and* $\mathbf{b} = 2\mathbf{j} - 3\mathbf{k}$.

Solution.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix}.$$

$$= \mathbf{i}[(-1)(-3) - (1)(2)] - \mathbf{j}[(1)(-3) - (1)(0)] + \mathbf{k}[(1)(2) - (-1)(0)].$$

$$= \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}.$$

Thus area is

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{1 + 9 + 4} = \sqrt{14}.$$

□

Example 17.5. *Prove by cross product that* $A(2, -3)$, $B(6, 4)$, $C(-2, -6)$ *and* $D(-6, -13)$ *are vertices of a parallelogram after ordering the points appropriately.*

Solution. In two dimensions, compare opposite side vectors. Compute

$$\overrightarrow{AB} = (6 - 2, 4 - (-3)) = (4, 7),$$

$$\overrightarrow{DC} = (-2 - (-6), -6 - (-13)) = (4, 7).$$

Thus $\overrightarrow{AB} = \overrightarrow{DC}$. Also

$$\overrightarrow{AD} = (-6 - 2, -13 - (-3)) = (-8, -10),$$

$$\overrightarrow{BC} = (-2 - 6, -6 - 4) = (-8, -10).$$

Therefore opposite sides are equal and parallel, so the points form a parallelogram. □

17.3 Coordinate Geometry

Example 17.6. *Find the angle between the lines* $2x - y + 3 = 0$ *and* $x + 3y - 4 = 0$.

Solution. For $2x - y + 3 = 0$, $y = 2x + 3$, so $m_1 = 2$. For $x + 3y - 4 = 0$, $y = -\frac{1}{3}x + \frac{4}{3}$, so $m_2 = -\frac{1}{3}$.

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{3} - 2}{1 + 2(-\frac{1}{3})}\right| = \left|\frac{-\frac{7}{3}}{\frac{1}{3}}\right| = 7.$$

Hence $\theta = \tan^{-1}(7)$. □

Example 17.7. *Find the tangent to the circle* $x^2 + y^2 = 25$ *at* $(3, 4)$.

Solution. For $x^2 + y^2 = r^2$, tangent at (x_1, y_1) is

$$xx_1 + yy_1 = r^2.$$

Thus

$$3x + 4y = 25.$$

□

17.4 Dynamics

Example 17.8. *A stone is thrown vertically upward with speed* 30 *m/s. Take* $g = 10$ *m/s*^{2}*. Find the time to reach maximum height and the maximum height.*

Solution. At maximum height, $v = 0$. Using $v = u - gt$,

$$0 = 30 - 10t \Rightarrow t = 3 \text{ s}.$$

Using $v^2 = u^2 - 2gs$,

$$0 = 30^2 - 2(10)s \Rightarrow 20s = 900 \Rightarrow s = 45 \text{ m}.$$

□

Chapter 18

Assignments

Assignment 1: Vectors and Linear Combination

1. Distinguish clearly between scalar and vector quantities, giving five examples of each.
2. If $A(2,-1,3)$ and $B(5,4,-2)$, find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.
3. Find a unit vector in the direction of $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
4. Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, compute $3\mathbf{a} - 2\mathbf{b}$.
5. Express $(4,7)$ as a linear combination of $(1,1)$ and $(2,3)$.
6. Determine whether $(1,2,3)$, $(2,1,0)$ and $(3,3,3)$ are linearly independent.

Assignment 2: Dot Product and Cross Product

1. Find the angle between $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
2. Find λ if $2\mathbf{i} - \mathbf{j} + \lambda\mathbf{k}$ is perpendicular to $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.
3. Compute $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$.
4. Find the area of the triangle with vertices $A(0,0,0)$, $B(1,1,1)$ and $C(0,2,3)$.
5. Find a unit vector perpendicular to $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

Assignment 3: Geometry and Dynamics

1. Find the equation of the line through $(1,3)$ and $(5,11)$.
2. Find the centre and radius of $x^2 + y^2 + 6x - 8y + 9 = 0$.
3. Find the tangent to $x^2 + y^2 = 50$ at $(5,5)$.
4. A body starts from rest with acceleration $2\,\mathrm{m/s^2}$. Find its velocity and distance after 10 seconds.

5. A projectile is fired at 40 m/s at an angle of 30°. Take $g = 10\,\mathrm{m/s^2}$. Find time of flight and range.

Chapter 19

Model Solutions to Assignments

Solutions to Assignment 1

1. Scalar quantities have magnitude only; vector quantities have magnitude and direction. Scalars: mass, time, speed, distance, temperature. Vectors: displacement, velocity, acceleration, force, momentum.

2. $\overrightarrow{AB} = (3, 5, -5) = 3\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ and $|\overrightarrow{AB}| = \sqrt{9 + 25 + 25} = \sqrt{59}$.

3. Magnitude is $\sqrt{36 + 4 + 9} = 7$, so unit vector is $\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$.

4. $3\mathbf{a} = 3\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$, $2\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, so $3\mathbf{a} - 2\mathbf{b} = -\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$.

5. Let $(4, 7) = a(1, 1) + b(2, 3)$. Then $a + 2b = 4$, $a + 3b = 7$. Subtracting gives $b = 3$, hence $a = -2$. Therefore $(4, 7) = -2(1, 1) + 3(2, 3)$.

6. Determinant:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix} = 1(3) - 2(6) + 3(3) = 3 - 12 + 9 = 0.$$

   They are linearly dependent.

Solutions to Assignment 2

1. $\mathbf{a} \cdot \mathbf{b} = 3 - 2 - 4 = -3$. $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{14}$. Hence $\cos\theta = -\frac{1}{\sqrt{14}}$.

2. Perpendicularity gives $2(1) + (-1)(3) + \lambda(-2) = 0$, so $-1 - 2\lambda = 0$, hence $\lambda = -\frac{1}{2}$.

3.
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix} = 2\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}.$$

4. $\overrightarrow{AB} = (1, 1, 1)$, $\overrightarrow{AC} = (0, 2, 3)$. Then $\overrightarrow{AB} \times \overrightarrow{AC} = (1, -3, 2)$. Area $= \frac{1}{2}\sqrt{1 + 9 + 4} = \frac{\sqrt{14}}{2}$.

5. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$. Unit normal is $\frac{1}{\sqrt{42}}(\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$.

Solutions to Assignment 3

1. Slope $m = (11 - 3)/(5 - 1) = 2$. Equation: $y - 3 = 2(x - 1)$, hence $y = 2x + 1$.
2. $x^2 + 6x + y^2 - 8y = -9$. Thus $(x + 3)^2 + (y - 4)^2 = 16$. Centre $(-3, 4)$, radius 4.
3. Tangent is $5x + 5y = 50$, so $x + y = 10$.
4. $v = u + at = 0 + 2(10) = 20\,\mathrm{m/s}$. $s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}(2)(100) = 100\,\mathrm{m}$.
5. $T = 2u \sin 30°/g = 2(40)(1/2)/10 = 4$ s. $R = u^2 \sin 60°/g = 1600(\sqrt{3}/2)/10 = 80\sqrt{3}$ m.

Chapter 20

Revision Test with Worked Solutions

Test Questions

1. Let $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.
2. Find the direction cosines of $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
3. Show that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ implies $\mathbf{a}$ is perpendicular to $\mathbf{b}$.
4. Find the area of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.
5. Find the equation of a circle with centre $(2, -1)$ passing through $(5, 3)$.
6. A particle has displacement $s = t^3 - 6t^2 + 9t$. Find its velocity and acceleration at $t = 2$.

Worked Solutions

1. $\mathbf{a} + \mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. $\mathbf{a} - \mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$. $\mathbf{a} \cdot \mathbf{b} = 3 - 8 - 2 = -7$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 4 & -2 \end{vmatrix} = 0\mathbf{i} + 7\mathbf{j} + 14\mathbf{k}.$$

2. Magnitude is $\sqrt{4 + 9 + 36} = 7$. Direction cosines are $2/7, -3/7, 6/7$.
3. Squaring both sides gives

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}).$$

Thus

$$|\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2.$$

Therefore $4\mathbf{a} \cdot \mathbf{b} = 0$, hence $\mathbf{a} \cdot \mathbf{b} = 0$. Thus they are perpendicular.

4. Take $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. Then $\overrightarrow{AB} = (-1, 1, 0)$ and $\overrightarrow{AC} = (-1, 0, 1)$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = (1, 1, 1).$$

Area $= \frac{1}{2}\sqrt{3}$.

5. Radius squared is $(5-2)^2 + (3+1)^2 = 9 + 16 = 25$. Equation:

$$(x-2)^2 + (y+1)^2 = 25.$$

6. $v = ds/dt = 3t^2 - 12t + 9$, so $v(2) = 12 - 24 + 9 = -3$. $a = dv/dt = 6t - 12$, so $a(2) = 0$.

Chapter 21

Additional Solved Examination Problems

This chapter supplies extra past-question style problems for MTH 121. Each problem is solved in a form suitable for class teaching and revision.

21.1 More Problems on Linear Combinations

Example 21.1. *Determine whether* $\mathbf{w} = (1, 5, 7)$ *belongs to the span of* $\mathbf{u} = (1, 1, 2)$ *and* $\mathbf{v} = (2, 3, 1)$.

Solution. We seek a, b such that

$$a(1, 1, 2) + b(2, 3, 1) = (1, 5, 7).$$

Equating components,

$$a + 2b = 1, \quad a + 3b = 5, \quad 2a + b = 7.$$

Subtracting the first equation from the second gives $b = 4$. Then $a + 8 = 1$, so $a = -7$. Check the third equation:

$$2(-7) + 4 = -10 \neq 7.$$

The equations are inconsistent. Therefore $\mathbf{w}$ is not in the span of $\mathbf{u}$ and $\mathbf{v}$. □

Example 21.2. *Find* p *such that* $\mathbf{a} = (1, p, 3)$, $\mathbf{b} = (2, 1, 4)$ *and* $\mathbf{c} = (3, 5, 7)$ *are linearly dependent.*

Solution. The vectors are linearly dependent when

$$\det \begin{pmatrix} 1 & p & 3 \\ 2 & 1 & 4 \\ 3 & 5 & 7 \end{pmatrix} = 0.$$

Expanding,

$$1 \begin{vmatrix} 1 & 4 \\ 5 & 7 \end{vmatrix} - p \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 0.$$

Thus

$$(7 - 20) - p(14 - 12) + 3(10 - 3) = 0.$$

$$-13 - 2p + 21 = 0 \Rightarrow 8 - 2p = 0 \Rightarrow p = 4.$$

□

Example 21.3. *Resolve* $\mathbf{a} = 10\mathbf{i} + 6\mathbf{j}$ *into components parallel and perpendicular to* $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$.

Solution. The projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}.$$

Now $\mathbf{a} \cdot \mathbf{b} = 10(3) + 6(4) = 54$ and $|\mathbf{b}|^2 = 3^2 + 4^2 = 25$. Hence

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{54}{25}(3\mathbf{i} + 4\mathbf{j}) = \frac{162}{25}\mathbf{i} + \frac{216}{25}\mathbf{j}.$$

The perpendicular component is

$$\mathbf{a} - \operatorname{proj}_{\mathbf{b}} \mathbf{a} = \left(10 - \frac{162}{25}\right)\mathbf{i} + \left(6 - \frac{216}{25}\right)\mathbf{j} = \frac{88}{25}\mathbf{i} - \frac{66}{25}\mathbf{j}.$$

□

21.2 More Problems on Lines and Circles

Example 21.4. *Find the equation of the line passing through* $(3, -2)$ *and perpendicular to* $4x - 2y + 7 = 0$.

Solution. Write the given line as

$$-2y = -4x - 7 \Rightarrow y = 2x + \frac{7}{2}.$$

Its slope is 2. A perpendicular line has slope $-\frac{1}{2}$. Through $(3, -2)$:

$$y + 2 = -\frac{1}{2}(x - 3).$$

Multiplying by 2 gives

$$2y + 4 = -x + 3 \Rightarrow x + 2y + 1 = 0.$$

□

Example 21.5. *Find the equation of the circle whose diameter has endpoints* $A(1, 2)$ *and* $B(7, 10)$.

Solution. The centre is the midpoint:

$$C = \left(\frac{1+7}{2}, \frac{2+10}{2}\right) = (4, 6).$$

The radius is half the distance AB:

$$AB = \sqrt{(7-1)^2 + (10-2)^2} = \sqrt{36 + 64} = 10.$$

Thus $r = 5$ and the circle is

$$(x - 4)^2 + (y - 6)^2 = 25.$$

□

Example 21.6. *Find the normal to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ at the point $(6, 2)$.*

Solution. Complete the square:

$$x^2 - 6x + y^2 + 4y = 12,$$

$$(x - 3)^2 + (y + 2)^2 = 25.$$

The centre is $C(3, -2)$. The normal at a point on a circle is the line joining the centre to that point. Slope of the normal through $C(3, -2)$ and $P(6, 2)$ is

$$m = \frac{2 - (-2)}{6 - 3} = \frac{4}{3}.$$

Equation through $(6, 2)$:

$$y - 2 = \frac{4}{3}(x - 6).$$

So

$$4x - 3y - 18 = 0.$$

□

21.3 More Problems on Mechanics

Example 21.7. *A particle moves in a straight line with acceleration $a = 6t - 4$. If $v = 3$ when $t = 0$ and $s = 2$ when $t = 0$, find v and s as functions of t.*

Solution. Since $a = dv/dt$,

$$\frac{dv}{dt} = 6t - 4.$$

Integrating,

$$v = 3t^2 - 4t + C.$$

Given $v(0) = 3$, we get $C = 3$. Hence

$$v = 3t^2 - 4t + 3.$$

Since $v = ds/dt$,

$$\frac{ds}{dt} = 3t^2 - 4t + 3.$$

Integrating,

$$s = t^3 - 2t^2 + 3t + C_1.$$

Given $s(0) = 2$, $C_1 = 2$. Thus

$$s = t^3 - 2t^2 + 3t + 2.$$

□

Example 21.8. *A body of mass 10 kg is pulled along a horizontal plane by a force of 50 N inclined at 30° above the horizontal. If the plane is smooth, find the horizontal acceleration.*

Solution. The horizontal component of the force is

$$F_x = 50\cos 30^\circ = 50 \cdot \frac{\sqrt{3}}{2} = 25\sqrt{3}.$$

Using $F = ma$ horizontally,

$$a = \frac{F_x}{m} = \frac{25\sqrt{3}}{10} = \frac{5\sqrt{3}}{2}\,\mathrm{m/s^2}.$$

□

Example 21.9. *A particle of mass 2 kg moves with velocity* $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ *m/s. Find its momentum and kinetic energy.*

Solution. Momentum is

$$\mathbf{p} = m\mathbf{v} = 2(3\mathbf{i} - 4\mathbf{j}) = 6\mathbf{i} - 8\mathbf{j}\,\mathrm{kg\ m/s}.$$

Speed is

$$|\mathbf{v}| = \sqrt{3^2 + (-4)^2} = 5\,\mathrm{m/s}.$$

Kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(2)(5^2) = 25\,\mathrm{J}.$$

□

Chapter 22

Additional Practice Questions Without Solutions

Vector Algebra

1. Find α and β such that $(8, 1, 10) = \alpha(1, 2, 3) + \beta(2, -1, 1)$, if possible.
2. Determine whether $(1, 0, 2)$, $(2, 1, 3)$ and $(3, 1, 5)$ are linearly independent.
3. Find the angle between $4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.
4. Find the projection of $2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ on $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
5. Find the area of the triangle with vertices $(2, 1, 0)$, $(1, 3, 2)$ and $(4, 0, 1)$.

Geometry and Dynamics

1. Find the equation of the line through $(2, 5)$ parallel to $3x + 4y - 1 = 0$.
2. Find the equation of the circle through $(1, 0)$, $(0, 1)$ and $(1, 2)$.
3. Find the range of a projectile fired at $25\,\mathrm{m/s}$ at $45°$ to the horizontal.
4. A force $5\mathbf{i} + 12\mathbf{j}$ N moves a particle through displacement $4\mathbf{i} - 3\mathbf{j}$ m. Find the work done.
5. If $x = 4\sin(3t)$, find the maximum speed in the SHM.

Chapter 23

Short Formula Summary

Topic	Formula
Magnitude	$\lvert x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\rvert = \sqrt{x^2 + y^2 + z^2}$
Unit vector	$\widehat{\mathbf{a}} = \mathbf{a}/\lvert\mathbf{a}\rvert$
Dot product	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$
Component dot product	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Cross product	$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
Area of parallelogram	$\lvert\mathbf{a} \times \mathbf{b}\rvert$
Area of triangle	$\frac{1}{2}\lvert\mathbf{a} \times \mathbf{b}\rvert$
Scalar triple product	$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
Line slope	$m = (y_2 - y_1)/(x_2 - x_1)$
Circle	$(x - a)^2 + (y - b)^2 = r^2$
Newton's second law	$\mathbf{F} = m\mathbf{a}$
Projectile range	$R = u^2 \sin 2\theta/g$
SHM	$a = -\omega^2 x$

Final Study Advice

For examination preparation, students should master definitions, then practise computations. In vector algebra, most errors occur in signs during determinant expansion and in confusing dot product with cross product. In dynamics, clearly choose the positive direction before substituting values into the equations of motion.

Final Remarks

These notes provide a complete teaching foundation for MTH 121. Students are advised to practise the worked examples repeatedly, master the determinant method for cross products, understand linear combinations and independence, and apply vector methods to geometry and mechanics.